($\phi_{h,\alpha}$) between the plunging and pitching motions at the elastic axis vs mass ratio are plotted for three typical Mach numbers. At $M_\infty = 0.825$, the phase difference rapidly decreases to zero with increasing mass ratio, while the amplitude ratio approaches to the constant value of 1.86 when μ is greater than 40. Since the zero phase difference between the plunging and pitching motions implies the existence of the pivotal point at $x_{pv} = a-(h/b)/\alpha$ (a is the elastic axis position behind midchord in percent semichord and $a = -2.0$ for the present system), the flutter mode at $M_\infty = 0.825$ for μ greater than 40 is essentially a pitching oscillation with pivotal point at $x_{pv} = -3.86$, which is almost identical to the first natural mode shown in Fig. 2. Therefore, when we recall the fact that the large negative peak value of the out-of-phase component in the load distribution at $M_\infty = 0.825$ is due to the phase lag of the shock wave motion, it is concluded that the shock wave (especially the phase delay of the shock-wave motion) is playing the dominant role in the mechanism of the transonic dip phenomenon.

It is believed that the conclusions reached in this investigation may shed some light on the mechanism of the transonic dip phenomenon of a sweptback wing flutter, although the important viscous and three-dimensional effects are still not taken into account in the present analysis.

References

[1]Isogai, K., "On the Transonic-Dip Mechanism of Flutter of a Sweptback Wing," *AIAA Journal,* Vol. 17, July 1979, pp. 793-795.

[2]Albano, E. and Rodden, W. P., "A Doublet-Lattice Method for Calculating Lift Distributions on Oscillating Surfaces in Subsonic Flows," *AIAA Journal,* Vol. 7, Feb. 1969, pp. 279-285.

[3]Ashley, H., "Role of Shocks in the 'Sub-Transonic' Flutter Phenomenon," *Journal of Aircraft,* Vol. 17, March 1980, pp. 187-197.

[4]Isogai, K., "Numerical Study of Transonic Flutter of a Two-Dimensional Airfoil," National Aerospace Laboratory, Japan, TR-617T, July 1980.

AIAA 81-4230

New Approach to the Solution of Falkner-Skan Equation

A. Aziz*
University of Riyadh, Saudi Arabia
and
T. Y. Na*
University of Michigan-Dearborn, Dearborn, Mich.

Introduction

THE Falkner-Skan equation is a celebrated equation in fluid mechanics. It describes the two-dimensional, incompressible boundary-layer flow over a wedge when the freestream velocity u_∞ is of the form $u_\infty \sim x^{(\beta/2-\beta)}$. In mathematical terms, it constitutes a third-order, nonlinear two-point boundary-value problem and is usually written as

$$f''' + ff'' + \beta[1-(f')^2] = 0 \tag{1}$$

$$f(0) = f'(0) = 0; \qquad f'(\infty) = 1 \tag{2}$$

Received Dec. 4, 1980; revision received March 6, 1981.

*Professor, Department of Mechanical Engineering.

Since no closed-form solution is known, a variety of numerical schemes have been used and solutions for a range of values of β tabulated. The standard shooting technique in which a succession of values of $f''(0)$ are guessed until $f'(\infty) = 1$ is satisfied has been used by Smith[1] and Elzy and Sisson.[2] However, a major difficulty arises for negative values of β. For example, with $\beta = -0.19$, three values of $f''(0) = 0.040$, 0.08570, and 0.130 satisfy the condition $f'(\infty) = 1$, of which the physically meaningful solution is the one with $f''(0) = 0.08570$.

In a later study, Cebeci and Keller[3] used Newton's method to refine the initial guess and found that with their scheme, the solution automatically converged to the one that was physically relevant. However, the convergence still depended on the initial guess. To circumvent this difficulty, a further refinement was introduced by employing the parallel shooting method in which the boundary layer was divided into three subintervals. This necessitated guessing of seven values, instead of one, though the sensitivity of the solution to initial guess was reduced significantly. Most recently, Zagustin et al.[4] adopted the quasilinearization scheme to obtain the solution. Being akin to Newton's method, it also has the disadvantage that a poor initial guess leads to divergence.

Other numerical efforts toward solving the Falkner-Skan equation have concentrated on using noniterative schemes such as parameter differentiation,[5,6] general parameter mapping,[7] and invariant imbedding.[6] Based on our own experience with the present problem, all three methods could be regarded comparable in computational effort and accuracy.

In the present work, we describe a new approach to solve the Falkner-Skan equation. Compared to the previously cited methods, the present method is much simpler both in concept and computation but yields results of high accuracy. To solve Eqs. (1) and (2), we first assume a regular perturbation expansion for f as a power series in β and generate a total of eleven terms. As is usual, attention is focused on the quantity $f''(0)$ which is a measure of wall shear. The resulting series for $f''(0)$ is found to converge for small values of β. However, the range of applicability and accuracy are remarkably improved by forming the partial sums and applying the Shanks transformation[8] five times. Indeed, for the entire range of physical interest, that is, $-0.19884 \leq \beta \leq 2$, the procedure yields values of $f''(0)$ correct to four decimal places for most values of β and to three decimal places for others, when compared to the solutions of Smith[1] and Cebeci and Keller.[3]

Solution Method

Considering Eqs. (1) and (2), we assume a regular perturbation expansion in β of the form

$$f = \sum_{n=0}^{\infty} \beta^n f_n \tag{3}$$

Substituting Eq. (3) into Eqs. (1) and (2), equating like coefficients of β, and truncating the expansion at the eleventh term, we have

$$\beta^0 : f_0''' + f_0 f_0'' = 0 \tag{4}$$

$$f_0(0) = f_0'(0) = 0; f_0'(\infty) = 1 \tag{5}$$

$$\beta^n : f_n''' + f_0 f_n'' + f_0'' f_n = \phi_n \tag{6}$$

$$f_n(0) = f_n'(0) = 0; f_n'(\infty) = 0 \quad (n = 1, 2, \ldots, 10) \tag{7}$$

where ϕ_n for successive terms in the series are

$$\phi_1 = (f_0')^2 - 1 \tag{8}$$

$$\phi_2 = 2f_0' f_1' - f_1 f_1'' \tag{9}$$

Table 1 Repeated application of Shanks transformation to series Eq. (18): $\beta = 1$

n	S_n	e	e^2	e^3	e^4	e^5
0	0.469600					
1	1.768529	1.067675				
2	0.246380	1.312898	1.210872			
3	3.809349	1.138180	1.241830	1.230621		
4	−6.862651	1.392997	1.224260	1.233572	1.232539	
5	29.599035	0.893491	1.244073	1.231982	1.232697	1.232623
6	−105.345958	2.069371	1.213054	1.233280	1.232560	
7	421.182851	−1.081582	1.271184	1.231661		
8	−1711.227061	8.206277	1.147714			
9	7167.089345	−21.201637				
10	−30595.609874					

Table 2 Comparison of values of $f''(0)$

	$f''(0)$		
β	Smith[1]	Cebeci and Keller[3]	Present
2.0	1.687218	–	1.687516
1.6	1.521514	1.521516	1.521689
1.2	1.335722	1.335724	1.335793
1.0	1.232588	1.232561	1.232623
0.8	1.120268	1.120269	1.120280
0.6	0.995836	–	0.995837
0.4	0.854421	0.854423	0.854418
0.2	0.686708	0.686711	0.686706
0.1	0.587035	0.587037	0.587034
0.05	0.531130	–	0.531129
0.00	0.469600	0.469603	0.469600
−0.05	0.400323	0.400330	0.400322
−0.10	0.319270	0.319278	0.319266
−0.14	0.239736	–	0.239724
−0.16	0.190780	–	0.190758
−0.18	0.128636	–	0.128615
−0.19	0.085700	0.085702	0.085840
−0.195	0.055172	0.055177	0.056027

$$\phi_3 = 2f_0'f_2' + (f_1')^2 - f_1f_2'' - f_2f_1'' \tag{10}$$

$$\phi_4 = 2(f_0'f_3' + f_1'f_2') - f_1f_3'' - f_2f_2'' - f_3f_1'' \tag{11}$$

$$\phi_5 = 2(f_0'f_4' + f_1'f_3') + (f_2')^2 - f_1f_4'' - f_2f_3'' - f_3f_2'' - f_4f_1'' \tag{12}$$

$$\phi_6 = 2(f_0'f_5' + f_1'f_4' + f_2'f_3') - f_1f_5'' - f_2f_4'' - f_3f_3'' - f_4f_2'' - f_5f_1'' \tag{13}$$

$$\phi_7 = 2(f_0'f_6' + f_1'f_5' + f_2'f_4') + (f_3')^2 - f_1f_6'' - f_2f_5'' - f_3f_4'' - f_4f_3'' - f_5f_2'' - f_6f_1'' \tag{14}$$

$$\phi_8 = 2(f_0'f_7' + f_1'f_6' + f_2'f_5' + f_3'f_4') - f_1f_7'' - f_2f_6'' - f_3f_5'' - f_4f_4'' - f_5f_3'' - f_6f_2'' - f_7f_1'' \tag{15}$$

$$\phi_9 = 2(f_0'f_8' + f_1'f_7' + f_2'f_6' + f_3'f_5') + (f_4')^2 - f_1f_8'' - f_2f_7'' - f_3f_6'' - f_4f_5'' - f_5f_4'' - f_6f_3'' - f_7f_2'' - f_8f_1'' \tag{16}$$

$$\phi_{10} = 2(f_0f_9' + f_1'f_8' + f_2'f_7' + f_3'f_6' + f_4'f_5') - f_1f_9'' - f_2f_8'' - f_3f_7'' - f_4f_6'' - f_5f_5'' - f_6f_4'' - f_7f_3'' - f_8f_2'' - f_9f_1'' \tag{17}$$

The zero-order problem, Eqs. (4) and (5), is recognized as the classical Blasius problem whose solution is well documented. Equations (6) and (7) constitute a set of linear boundary-value problems whose solution can be obtained noniteratively by the method of superposition.[6]

Since the quantity of primary interest is the wall shear which is proportional to $f''(0)$, we concentrate on the series for $f''(0)$. From the numerical solutions the eleven-term series for $f''(0)$ can be written as

$$f''(0) = 0.46960 + 1.29893\beta - 1.52215\beta^2 + 3.56297\beta^3 - 10.6720\beta^4 + 36.4617\beta^5 - 134.945\beta^6 + 526.529\beta^7 - 2132.41\beta^8 + 8878.32\beta^9 - 37762.7\beta^{10} \tag{18}$$

Since the range of physical interest is $-0.19884 \leq \beta \leq 2$, it is obvious that Eq. (18) as it stands cannot cover this range. In fact, it is useful only for small values of β.

To improve the series Eq. (18) we apply Shanks transformation[8] to the partial sums. A sample application for $\beta = 1$ is shown in Table 1. The second column gives the partial sum S_n while the result of Shank transformation $e(S_n)$ appears in the third column where the operator e is defined as[8]

$$e(S_n) = \frac{S_{n+1}S_{n-1} - S_n^2}{S_{n+1} + S_{n-1} - 2S_n} \tag{19}$$

The preceding transformation is applied repeatedly and the results shown in the subsequent columns. It is interesting to note that the partial sums S_n which oscillate widely are progressively smoothed out as the transformation is repeated. The last column gives the final value of $f''(0) = 1.232623$ which is in excellent agreement with the values of 1.232588 of Smith[1] and 1.232561 of Cebeci and Keller.[3]

The final results are given in Table 2 for $-0.19884 \leq \beta \leq 2$. For comparison, the results of Smith[1] and Cebeci and Keller[3] are also included. The present results agree at least to three decimal places for all values of β in the aforementioned range.

Concluding Remarks

The present contribution is a remarkably successful application in fluid mechanics of the method of computer extension of perturbation series and subsequent improvement with Shanks transformation. Judged against the computational sophistication and effort previously required for the problem,[3] the present approach stands distinctly superior. The motivation for the present approach came from Van Dyke[8] who used it to improve the series for ground-state energy of anharmonic oscillator. Besides Shanks transformation, other techniques of improvement such as Euler transformation, Pade approximants, series reversion, extraction of singularity also have been used successfully both in fluid mechanics[8] and in heat transfer.[9] A very recent paper by Curle[10] provides another interesting application of the method to a boundary-layer problem.

References

[1]Smith, A. M. O., "Improved Solutions of Falkner-Skan Boundary Layer Equation," Fund Paper, *Journal of the Aerospace Sciences,* Sherman M. Fairchild, 1954.

[2]Elzy, E. and Sisson, R. M., "Tables of Similar Solutions to the Equations of Momentum, Heat and Mass Transfer in Laminar Boundary Layer Flow," Engineering Experiment Station, Oregon State Univ., Corvallis, Ore., Bulletin 40, 1967.

[3]Cebeci, T. and Keller, H. B., "Shooting and Parallel Shooting Methods for Solving the Falkner-Skan Boundary Layer Equation," *Journal of Computational Physics,* Vol. 7, 1971, pp. 289-300.

[4]Zagustin, E., Kalaba, R., and Ikeda, G., "Solution of the Falkner-Skan Equation by Quasilinearization," *Proceedings of the First Conference on Numerical Methods in Laminar and Turbulent Flow,* edited by C. Taylor, K. Morgan, and C. A. Brebbia, Pentech Press, London, 1978, pp. 379-385.

[5]Rubbert, P. E. and Landahl, M. T., "Solution of Nonlinear Flow Problems through Parameter Differentiation," *Physics of Fluids,* Vol. 10, No. 4, 1967, pp. 831-835.

[6]Na, T. Y., *Computational Methods in Engineering Boundary Value Problems,* Academic Press, 1979.

[7]Na, T. Y., "General Parameter Mapping of the Falkner-Skan Equation," *11th Annual Meeting of Society of Engineering Science,* Duke University, N. C., Paper 11, Nov. 11-13, 1974.

[8]Van Dyke, M., "Analysis and Improvement of Perturbation Series," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 27, 1974, pp. 423-450.

[9]Aziz, A., "Analysis of Heat Transfer Problems with Computer Extended Series," *Numerical Heat Transfer,* Vol. 4, No. 1, 1981, pp. 123-130.

[10]Curle, S. N., "An Unsteady Boundary Layer Problem—A Critical Test of Extended Series Expansions," *Journal of the Institute of Mathematical Applications,* Vol. 25, 1980, pp. 199-209.

AIAA 81-4231

Radiative Heat Transfer in the Presence of Obscurations

Leib Finkelstein* and Y. Weissman†
Soreq Nuclear Research Center, Yavne, Israel

I. Introduction

THE radiative heat transfer problem, in axisymmetric geometry and in the presence of an active gas, frequently occurs in the thermal design of aircrafts and spacecrafts, and as such it was considered by many authors (see, e.g., Refs. 1-7). In a nonparticipating medium, an important part of this calculation reduces to the shape factor. For complicated and/or multilayered axisymmetric bodies the approaches mentioned are of limited practical value, since the problem must then be solved numerically. This requires a subdivision of the whole body into a large number of surface elements, and the calculation of the radiative heat exchange between each pair of them. One is immediately faced with the complication introduced by the presence of full or partial obscurations produced by the various parts of the body. Though it is often the most time-consuming part of the whole calculation, it did not seem to have received a systematic treatment. The present Note aims to close this gap. It gives a short account of a conceptually different approach for the resolution of obscurations, and summarizes the corresponding algorithm. The method was tested, and it was utilized in a comprehensive computer program. Further details on the algorithm and on its incorporation in a general radiative heat transfer scheme is found in our report.[8]

Received March 24, 1980; revision received March 9, 1981.

*Associate Research Professor.

†Senior Research Scientist.

II. Obscurations in Axisymmetric Bodies

We start by analyzing the elementary situation shown in Fig. 1. The circles $S(x_\ell)$ and $S(x_m)$ are sections of a radiating surface at $x=x_\ell$ and $x=x_m$, with radii R_ℓ and R_m, respectively. The ring $S(x)$ is a cross section at x of an opaque layer formed by revolution of two profiles around the x axis; $r_i(x)$ is the running radius of the inner profile, while $r_e(x)$ is the running radius of the exterior profile. Consider the line of sight between a point P_m of $S(x_m)$, and P_ℓ, located at the bottom of $S(x_\ell)$. Let P be the point of intersection of $P_\ell P_m$ with the plane of $S(x)$, and let R be the distance of P to the axis. Denote by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ three mutually orthogonal coordinate unit vectors; and by $\boldsymbol{u}$, $\boldsymbol{u}_m$ and $\boldsymbol{\omega}$, unit vectors, pointing respectively in the $\boldsymbol{xP}$, $\boldsymbol{x_m P_m}$, and $\boldsymbol{P_\ell P}$ directions. Let further q and q_m be the lengths of $P_\ell P$ and of $P_\ell P_m$. The vision between P_ℓ and P_m is obstructed by $S(x)$ when P is located within the bounds of $S(x)$, that is, when R satisfies

$$r_i(x) < R < r_e(x) \tag{1a}$$

This will be transformed by the use of the following vector equations, deducible from Fig. 1,

$$\boldsymbol{i}(x-x_\ell) + \boldsymbol{u}R = -\boldsymbol{j}R_\ell + \boldsymbol{\omega} q$$

$$\boldsymbol{i}(x_m-x_\ell) + \boldsymbol{u}_m R_m = -\boldsymbol{j}R_\ell + \boldsymbol{\omega} q_m$$

$$\boldsymbol{u}_m = \boldsymbol{k}\cos[\phi-(\pi/2)] + \boldsymbol{j}\cos(\pi-\phi)$$

After some algebra we arrive at

$$\mu_{-1}(x) < \mu < \mu_1(x) \tag{1b}$$

where

$$\mu = \cos\phi, \quad \mu_{-1}(x) = \max\{-1,\ C[r_i(x)]\}$$

$$\mu_1(x) = \min\{1, C[r_e(x)]\}$$

$$C[r(x)] = \frac{[(x_m-x_\ell)r(x)]^2 - [(x_m-x)R_\ell]^2 - [(x-x_\ell)R_m]^2}{2[(x_m-x)R_\ell][(x-x_\ell)R_m]} \tag{2}$$

and $r(x)$ is either $r_i(x)$ or $r_e(x)$. Angles ϕ, satisfying both inequalities, correspond to lines of sight $P_\ell P_m$, obscured by the section $S(x)$.

The interval $[\mu_{-1}(x),\ \mu_1(x)]$ will be referred to as the interval of obscuration between $S(x_\ell)$ and $S(x_m)$, produced by the section $S(x)$. We shall extend the preceding argument to an opaque, smoothly changing layer occupying an interval $a \le x \le b$, between x_ℓ and x_m. Since the ring $S(x)$ is a section of the layer at x, it is evident that the domain of obscuration between $S(x_\ell)$ and $S(x_m)$, produced by the whole layer, is equal to the union of intervals $\bigcup_{a\le x\le b}[\mu_{-1}(x),\ \mu_1(x)]$. As x sweeps out $[a, b]$, the corresponding intervals of obscuration $[\mu_{-1}(x),\ \mu_1(x)]$, change continuously. It follows that the whole union is a single interval, say (μ_{-1}, μ_1). Thus,

$$\mu_{-1} = \min_{a\le x\le b} \mu_{-1}(x), \quad \mu_1 = \max_{a\le x\le b} \mu_1(x) \tag{3}$$

are the limits of the interval of obscuration produced by the whole layer.